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The Matrix Transformations on Double Sequence Space of χ^2_π

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Abstract. Let χ^2 denote the space of all prime sense double gai sequences and Λ^2 the space of all prime sense double analytic sequences. First we show that the set $E = \left\{ s^{(mn)} : m, n = 1, 2, 3, \cdots \right\}$ is a determining set for χ^2_π. The set of all finite matrices transforming χ^2_π into FK-space Y denoted by $\left(\chi^2_\pi : Y\right)$. We characterize the classes $\left(\chi^2_\pi : Y\right)$ when $Y = c^2_0, c^2, \chi^2, \ell^2, \Lambda^2$.

$\nearrow$	c^2_0	c^2	χ^2_π	ℓ^2	Λ^2
χ^2_π	Necessary and sufficient condition on the matrix are obtained				

But the approach to obtain these result in the present paper is by determining set for χ^2_π. First, we investigate a determining set for χ^2_π and then we characterize the classes of matrix transformations involving χ^2_π and other known sequence spaces.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$ the set of positive integers. Then w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on it was investigated by Hardy [3], Moricz [4], Moricz and Rhoades [5], Basarir and Solankan [1], Tripathy [6], Colak and Turkmenoglu [7], Turkmenoglu [8], and many others. We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$is called convergent if and only if the double sequence. (s_{mn}) is called convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, \dots)$ (see [9]). A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} \left|x_{mn}\right|^{1/m+n} < \infty$. The vector space of all double analytic sequences

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will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{\text{all finite sequences}\}$. Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$,

$$\Im_{mn} = \begin{pmatrix} 0, & 0, & \cdots & 0, & 0, & \cdots \\ 0, & 0, & \cdots & 0, & 0, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0, & 0, & \cdots & \pi_{mn}, & -\pi_{mn}, & \cdots \\ 0, & 0, & \cdots & 0, & 0, & \cdots \end{pmatrix}$$

with π_{mn} in the $(m,n)^{th}$ position, $-\pi_{mn}$ in the $(m+1, n+1)^{th}$ position and zero other wise. An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous. If X is a sequence space, we give the following definitions:

(i) $X^{'}$ = the continuous dual of X;
(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\}$;
(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X\right\}$;
(iv) $X^\gamma = \left\{a = (a_{mn}) : \overset{sup}{m, n \geq 1} \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$;
(v) let X be an FK-space $\supset \phi$; then $X^f = \{f(\Im_{mn}) : f \in X'\}$;
(vi) $X^\Lambda = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\right\}$

$X^\alpha X^\beta, X^\gamma$ are called $\alpha-$(or Köthe-Toeplitz) dual of X; $\beta-$(or generalized-Köthe-Toeplitz) dual of X; $\gamma-$dual of X, $\Lambda-$dual of X respectively.

2. DEFINITIONS AND PRELIMINARIES

$\chi^2_\pi = \left\{x = (x_{mn}) : \left(\frac{x_{mn}}{\pi_{mn}}\right) \in \chi^2\right\}$;
$\Lambda^2_\pi = \left\{x = (x_{mn}) : \left(\frac{x_{mn}}{\pi_{mn}}\right) \in \Lambda^2\right\}$.

The space Λ^2_π is a metric space with the metric

$$d(x,y) = sup_{mn}\left\{\left|\frac{x_{mn} - y_{mn}}{\pi_{mn}}\right|^{1/m+n} : m, n : 1, 2, 3, \ldots\right\} \tag{2}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Λ^2.

The space χ^2_π is a metric space with the metric

$$d(x,y) = sup_{mn}\left\{\left((m+n)!\left|\frac{x_{mn} - y_{mn}}{\pi_{mn}}\right|\right)^{1/m+n} : m, n : 1, 2, 3, \ldots\right\} \tag{3}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in χ^2.

Let X be an BK-space. Then $D = D(X) = \{x \in \phi : \|x\| \leq 1\}$ we do not assume that $X \supset \phi$ (i.e.) $D = \phi \bigcap (\text{ unit closed sphere in } X)$.

Let X be an BK space. A subset E of ϕ will be called a determining set for X if $D(X)$ is the absolutely convex hull of E. In respect of a metric space (X, d), $D = \{x \in \phi : d(x, 0) \leq 1\}$.

Given a sequence $x = \{x_{mn}\}$ and an four dimensional infinite matrix $A = \left(a_{mn}^{jk}\right), m, n, j, k = 1, 2, \ldots$ then $A-$ transform of x is the sequence $y = (y_{mn})$ when $y_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^{jk} x_{mn} \,(j, k = 1, 2, \ldots)$. Whenever $\sum \sum a_{mn}^{jk} x_{mn}$ exists.

Let X and Y be FK-spaces. If $y \in Y$ whenever $x \in X$, then the class of all matrices A is denoted by $(X : Y)$.

Lemma 2.1. *Let X be a BK-space and E is determining set for X. Let Y be an FK-space and A is an four dimensional infinite matrix. Suppose that either X has AK or A is row finite. Then $A \in (X : Y)$ if and only if (1) The columns of A belong to Y and (2) $A[E]$ is a bounded subset of Y.*

3. MAIN RESULTS

Theorem 3.1. *Let E be the set of all sequences in ϕ each of whose non-zero terms is*

$$\begin{pmatrix} 0, & 0, & \cdots & 0, & 0, & \cdots \\ 0, & 0, & \cdots & 0, & 0, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0, & 0, & \cdots & \frac{\pi_{mn}}{(m+n)!}, & \frac{-\pi_{mn}}{(m+n)!}, & \cdots \\ 0, & 0, & \cdots & 0, & 0, & \cdots \end{pmatrix}$$

with $\frac{\pi_{mn}}{(m+n)!}$, in the $(m,n)^{th}$, $\frac{-\pi_{mn}}{(m+n)!}$, in the $(m+1, n+1)^{th}$ position and zero other wise. Then E is determining set of χ_π^2.

Proof. **Step 1.** Recall that χ_π^2 is a metric space with the metric

$$d(x, y) = sup_{mn} \left\{ \left((m+n)! \left| \frac{x_{mn} - y_{mn}}{\pi_{mn}} \right| \right)^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}$$

Let A be the absolutely convex hull of E. Let $x \in A$.

Then $x = \sum_{m=1}^{i} \sum_{n=1}^{j} t_{mn} \pi_{mn} s^{(mn)}$ with

$$\sum_{m,n=1}^{i,j} |t_{mn}| \leq 1. \tag{4}$$

and $s^{(mn)} \in E$.

Then $d(x, 0) \leq |t_{11}| \pi_{11} d\left(s^{(11)}, 0\right) + \cdots + |t_{ij}| \pi_{ij} d\left(s^{(ij)}, 0\right)$. But $d\left(s^{(mn)}\right) = 1$ for $m, n = 1, 2, 3, \ldots, (i, j)$. Hence $d(x, 0) \leq \sum_{m,n=1}^{i,j} |t_{mn}| \leq 1$ by using

(4). Also $x \in \phi$. Hence $x \in D$. Thus

$$A \subset D \tag{5}$$

Step 2: Let $x \in D$

$$\Rightarrow x \in \phi \quad \text{and} \quad d(x,0) \le 1.$$

$$x = \begin{pmatrix} 2!x_{11}, & 3!x_{12}, & \cdots & (1+n)!x_{1n}, & \cdots \\ 3!x_{21}, & 4!x_{22}, & \cdots & (2+n)!x_{2n}, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ (m+1)!x_{m1}, & (m+2)!x_{m2}, & \cdots, & (m+n)!x_{mn}, & \cdots \\ 0, & 0, & \cdots & 0, & \cdots \end{pmatrix}$$

and

(6)

$$\sup \begin{pmatrix} (2!\,|x_{11}|)^{1/2}, & (3!\,|x_{12}|)^{1/3}, & \cdots & ((1+n)!\,|x_{1n}|)^{1/1+n}, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ ((m+1)!\,|x_{m1}|)^{1/m+1}, & ((m+2)!\,|x_{m2}|)^{1/m+2}, & \cdots & ((m+n)!\,|x_{mn}|)^{1/m+n}, & \cdots \\ 0, & 0, & \cdots & 0, & \cdots \end{pmatrix}$$

Case (i): Suppose that $2!\,|x_{11}| \ge \cdots \ge (m+n)!\,|x_{mn}|$.

Let $\xi_{mn} = Sgn\left((m+n)!x_{mn}\right) = \frac{(m+n)!|x_{mn}|}{(m+n)!x_{mn}}$ for $m,n = 1,2,\ldots,(i,j)$. Take

$$S_{k\ell}\pi_{k\ell} = \begin{pmatrix} \xi_{11}, & \xi_{12}, & \cdots & \xi_{1\ell}, & \cdots \\ \xi_{21}, & \xi_{22}, & \cdots & \xi_{2\ell}, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \xi_{k1}, & \xi_{k2}, & \cdots & \xi_{k\ell}, & \cdots \\ 0, & 0, & \cdots & 0, & \cdots \end{pmatrix}$$

for $k,\ell = 1,2,3,\ldots,(i,j)$.

Then $\pi_{k\ell}s_{k\ell} \in E$ for $k,\ell = 1,2,3,\ldots,(i,j)$.

Also

$$\begin{aligned} x = {} & \left(|2!x_{11} - 3!x_{12}| - |3!x_{21} - 4!x_{22}|\right)\pi_{11}S_{11} + \cdots \\ & +(|(m+n)!x_{mn} - (m+n+1)!x_{mn+1}| \\ & - |(m+n+1)!x_{m+1n} - (m+n+2)!x_{m+1n+1}|)\pi_{mn}S_{mn} \\ = {} & t_{11}\pi_{11}S_{11} + \cdots + t_{mn}\pi_{mn}S_{mn}, \end{aligned}$$

so that

$$\begin{aligned} t_{11} + \cdots + t_{mn} = {} & |2!x_{11} - 3!x_{12}| \\ & - |(m+n+1)!x_{m+1n} - (m+n+2)!x_{m+1n+1}| = |2!x_{11} - 3!x_{12}| \end{aligned}$$

because

$$|(m+n+1)!x_{m+1n} - (m+n+2)!x_{m+1n+1}| = 0 \le 1$$

by using (6).

Hence $x \in A$. Thus $D \subset A$.

Case (ii): Let y be x and let $2!\,|y_{11}| \geq \cdots \geq (m+n)!\,|y_{mn}|$. Express y as a member of A as in Case (i). Since E is invariant under permutation of the terms of its members, so is A. Hence $x \in A$. Thus $D \subset A$. Therefore in both cases

$$D \subset A \tag{7}$$

From (5) and (7) $A = D$. Consequently E is a determining set for χ^2_π. This completes the proof. □

Proposition 3.1. *χ^2_π has AK.*

Proof. Let $x = (x_{mn}) \in \chi^2_\pi$ and take $x^{[mn]} = \sum_{i,j=1}^{m,n} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$. Hence $d\left(x, x^{[rs]}\right) = sup_{mn}\left\{\left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}}\right|\right)^{1/m+n} : m \geq r+1, n \geq s+1\right\}$ $\to 0$ as $m,n \to \infty$

Therefore, $x^{[rs]} \to x$ as $r,s \to \infty$ in χ^2_π. Thus χ^2_π has AK. This completes proof. □

Proposition 3.2. *An infinite matrix $A = \left(a^{jk}_{mn}\right)$ is in the class*

$$A \in \left(\chi^2_\pi : c^2_0\right) \Leftrightarrow lim_{n,k\to\infty}\left(\pi_{mn}a^{jk}_{mn}\right) = 0 \tag{8}$$

$$\Leftrightarrow sup_{mn}\left|\pi_{m1}a^{j1}_{m1} + \cdots + \pi_{mn}a^{jk}_{mn}\right| < \infty. \tag{9}$$

Proof. In Lemma 3 take $X = \chi^2_\pi$ has AK property and take $Y = \left(c^2_0\right)$ be an FK-space. Further more χ^2_π is a determining set E (as in given Proposition 4). Also $A[E] = A\left(s^{(mn)}\right) = \left\{\left(\pi_{m1}a^{j1}_{m1} + \cdots + \pi_{mn}a^{jk}_{mn}\right)\right\}$. Again by Lemma 3, $A \in \left(\chi^2_\pi : c^2_0\right)$ if and only if:

(i) The columns of A belong to c^2_0, and

(ii) $A\left(s^{(mn)}\right)$ is a bounded subset χ^2_π.

But the condition

(i) $\Leftrightarrow \left\{\pi_{mn}a^{jk}_{mn} : j,k = 1,2,\cdots\right\}$ is exits for all m,n;

(ii) $\Leftrightarrow sup_{mn}\left|\pi_{m1}a^{j1}_{m1} + \cdots + \pi_{mn}a^{jk}_{mn}\right| < \infty$.

Hence we conclude that $A \in \left(\chi^2_\pi : c^2_0\right) \Leftrightarrow$ conditions (8) and (9) are satisfied. □

The following proofs are similar. Hence we omit the proof.

Proposition 3.3. *An infinite matrix $A = \left(a^{jk}_{mn}\right)$ is in the class*

$$A \in \left(\chi^2_\pi : c^2\right) \Leftrightarrow lim_{n,k\to\infty}\left(\pi_{mn}a^{jk}_{mn}\right)\, exists(m,j = 1,2,3,\ldots) \tag{10}$$

$$\Leftrightarrow sup_{mn} \left| \pi_{m1} a^{j1}_{m1} + \cdots + \pi_{mn} a^{jk}_{mn} \right| < \infty. \tag{11}$$

Proposition 3.4. *An infinite matrix* $A = \left(a^{jk}_{mn}\right)$ *is in the class*

$$A \in \left(\chi^2_\pi : \chi^2_\pi\right) \Leftrightarrow sup_{mn} \left(\frac{1}{\pi_{mn}(m+n)!} \left| a^{j1}_{m1} + \cdots + a^{jk}_{mn} \right| \right)^{1/m+n} < \infty. \tag{12}$$

$$\Leftrightarrow lim_{n,k\to\infty} \left(\frac{1}{\pi_{mn}(m+n)!} \left| a^{jk}_{mn} \right| \right)^{1/m+n} = 0, \ \textit{for } m, j = 1, 2, 3, \ldots \tag{13}$$

$$\Leftrightarrow d\left(a^{j1}_{m1}, a^{j2}_{m2}, \cdots, a^{jk}_{mn}\right) \ \textit{is bounded} \tag{14}$$
for each metric d on χ^2_π and for all m, n.

Proposition 3.5. *An infinite matrix* $A = \left(a^{jk}_{mn}\right)$ *is in the class*

$$A \in \left(\chi^2_\pi : \ell^2\right) \Leftrightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| a^{jk}_{mn} \right| \ \textit{converges} \ (j, k = 1, 2, 3, \ldots) \tag{15}$$

$$\Leftrightarrow sup_{mn} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \pi_{mn} a^{jk}_{mn} \right| < \infty \tag{16}$$

Proposition 3.6. *An infinite matrix* $A = \left(a^{jk}_{mn}\right)$ *is in the class*

$$A \in \left(\chi^2_\pi : \Lambda^2\right) \Leftrightarrow sup_{mn} \left(\left| \pi_{mn} \sum_{\gamma=1}^{n} \sum_{\mu=1}^{k} a^{j\mu}_{m\gamma} \right|^{1/m+n} \right) < \infty \tag{17}$$

$$\Leftrightarrow d\left(a^{j1}_{m1}, a^{j2}_{m2}, \cdots a^{jk}_{mn}\right) \quad \textit{is bounded} \tag{18}$$
for each metric d on Λ^2 and for all m, n.

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